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Nth-order minimum uncertainty products for arbitrary $N > -1$

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Abstract. This paper considers the problem of finding the quantum states that minimize the products of the 'absolute value' Nth-order fluctuation of two canonically conjugate operators. A scheme is applied to the problem, wherein the exact solution state is approximated by working in a subspace consisting of the ground state plus a finite number of higher-order states, using a variational technique. The solution is first obtained for the ground state, and this is subsequently compared to the solution assuming 2×2 and 3×3 spaces. For positive N we obtain a small, smooth decrease in the value of the fluctuation as the space increases and expect the 3×3 space results to be close to the exact Nth-order fluctuation. For negative N, however, we get an abrupt decrease in the fluctuation as the space increases. This result we interpret as being due to the fact the variational technique is trying to admix higher states (which have smaller components at the origin) into the wavefunction in order to counter the singularity of the $|x|^N$, $|p|^N$ operators at the origin.

1. Introduction

Recently the authors have considered [1] the problem of finding the states which minimize the product of the Nth-order fluctuations of two canonically conjugate operators. The Nth-order fluctuation of an operator is based on a simple generalization of the mean-squared fluctuation of an operator as defined in ordinary quantum mechanics, e.g., for an operator x, the Nth-order fluctuation is given by $(\Delta x)^N = \langle x^N\rangle - \langle x\rangle^N$, where $\langle\ \rangle$ denotes expectation value. The problem originated in connection with 'higher order squeezing' [2] and, as pointed out by Hong and Mandel who introduced the concept, in this context odd-N states have undesirable properties. Thus we restricted our attention to the even (non-trivial) integral values of $N = 2, 4, 6, \ldots$.

Since the appearance of [1] the authors have received a suggestion [3] that the work's limited, even-integral domain of N values could readily be widened by redefining the Nth-order fluctuation in terms of absolute values of the operators. Hence, letting U_N be the product of the (generalized) fluctuations of two canonically conjugate operators, x, p (with commutator $[x, p] = i$), we would now seek to minimize

$$U_N = \langle\Psi||x|^N|\Psi\rangle\langle\Psi||p|^N|\Psi\rangle. \tag{1}$$

In considering an expression such as (1) one may wonder how to evaluate the part containing the expectation value of the absolute value of an operator, e.g., $|p|$ in the position representation. The authors have adopted the following pragmatic technique. Given a wavevector $|\Psi\rangle$, go to the position representation where the operator x is represented by the c-number function x, the wavefunction is $\langle x|\Psi\rangle$, and $\langle\Psi||x|^N|\Psi\rangle$ is readily computable by the ordinary techniques of integral calculus. Now switch to the

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momentum representation wherein p is a c-number and with $\langle p|\Psi\rangle$ in hand (e.g., from the Fourier transform of $\langle x|\Psi\rangle$), $\langle\Psi||p|^N|\Psi\rangle$ may also be readily computed, and U_N found. Since the expectation value of an operator is independent of representation the result is the same as would be found by any other method. Thus (1) has been given unambiguous status.

The procedure used to minimize U_N is similar to that followed in [1]. U_N is computed when the wavevector is a trial function composed of a superposition of harmonic oscillator states, $|n\rangle$. In the course of the calculation it is found that only the states $n = 0, 4, 8, \ldots$ contribute. The coefficients of the oscillator states are varied and the minimum of U_N sought. The calculation can be done analytically for a trial wavevector containing only the states $|0\rangle$, $|4\rangle$. Once the state $|8\rangle$ is mixed in calculations must be carried out numerically, and this has been accomplished using the software program MathCAD [4]. Before turning to the details of the general calculation it is interesting to consider the values for U_N in the state $|0\rangle$.

2. Ground state value of U_N

As is well known the ground state of the harmonic oscillator minimizes the ordinary mean-squared fluctuation $\langle\Psi|(\Delta x^2)|\Psi\rangle\langle\Psi|(\Delta p^2)|\Psi\rangle$. Here

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$$

and similarly for p. It is thus of interest to consider U_N when $|\Psi\rangle = |0\rangle$.

The normalized ground state harmonic oscillator wavefunction in the position representation is [5]

$$\langle x|0\rangle = (1/\pi)^{1/4}\exp(-x^2/2) \tag{2}$$

with the corresponding wavefunction in the momentum representation of

$$\langle p|0\rangle = (1/2\pi)^{1/2}\int_{-\infty}^{\infty}\exp(ipx)\langle x|0\rangle\,dx = (1/\pi)^{1/4}\exp(-p^2/2). \tag{3}$$

The calculation of the product of the fluctuations is elementary, and one finds

$$[\langle\Psi||x|^N|\Psi\rangle\langle\Psi||p|^N|\Psi\rangle]^{1/2} = \Gamma(N)/2^{N-1}\Gamma(N/2) \tag{4}$$

where $\Gamma(N)$ is the gamma function. For $N = 2$ one obtains the value of $\frac{1}{2}$, the same as is found for the ordinary product of mean-squared fluctuations. A plot of the right-hand side of (4) is shown in figure 1. It is interesting that the $N = 2$ value of this function almost, but not quite, represents its absolute minimum. The actual absolute minimum has been found numerically to be at $N = 1.9233$, with the value 0.4996.

3. Solution for U_N

In [1] we derived a method for obtaining the minimum value for U_N. This yields the eigenvalue equation

$$\tfrac{1}{2}(|x|^N + |p|^N)|\Psi\rangle = \lambda|\Psi\rangle \tag{5}$$

where the wavefunction $|\Psi\rangle$ is normalized. The resulting eigenvalue λ is $(U_N)^{1/2}_{\min}$.

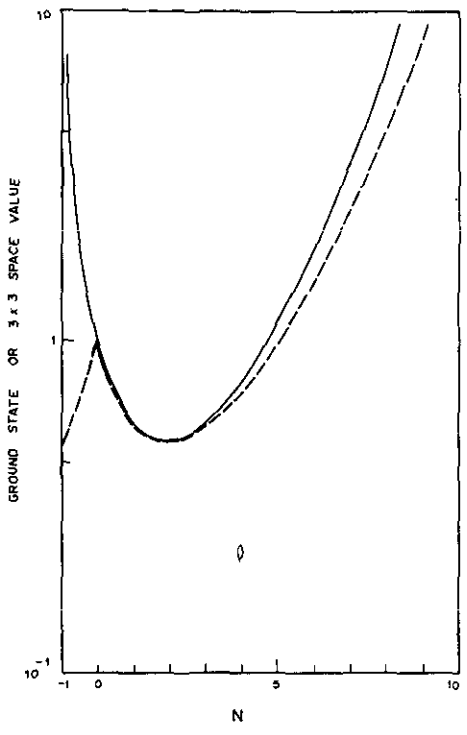

Figure 1. Plot of ground state solution (solid line). Plot of 3×3 solution (dotted line).

Define

$$H \equiv \tfrac{1}{2}(|x|^N + |p|^N). \tag{6}$$

Previously we broke this down as $H = H_0 + H_1$ and used the eigenfunctions of H_0 for our basis. Here we simplify our procedure and merely adopt the complete simple harmonic oscillator basis $|n\rangle$, expand $|\Psi\rangle = \Sigma_n c_n|n\rangle$, and solve (5) in successively larger truncated $K \times K$ spaces, using a numerical variational technique.

4. Exact solution in subspace spanned by $|0\rangle, |4\rangle$

We find that only $|0\rangle, |4\rangle, |8\rangle, \ldots$ contribute to the calculation. That the states $|1\rangle, |3\rangle, \ldots |5\rangle \ldots$ do not contribute follows from the fact that the Hamiltonian in (6) commutes with the parity operator. The reason why the states $|2\rangle$, $|6\rangle$, $|10\rangle, \ldots$ etc do not contribute is more subtle. This result can be obtained from general symmetry and group-theoretical arguments [3]. But it can also be understood by noting for instance that whereas $\langle p|0\rangle$ in (3) can be obtained from $\langle x|0\rangle$ in (2) (and similarly $\langle p|4\rangle \ldots$ can be obtained from $\langle x|4\rangle \ldots$) merely by replacing x by p, to go from $\langle x|2\rangle$ to $\langle p|2\rangle$ (or $\langle x|6\rangle \ldots$ to $\langle p|6\rangle$ etc.) also involves an additional minus sign. Thus

$$\langle x|2\rangle = \left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)^{1/4} e^{-x^2/2}(2x^2-1)$$

$$\langle p|2\rangle = -\left(\frac{1}{2}\right)^{1/2}\left(\frac{1}{\pi}\right)^{1/4} e^{-p^2/2}(2p^2-1)$$

3×3 SPACE CALCULATIONS

Set: $N := 5$

1. DEFINE MATRIX H

Matrix is H; define matrix elements:

$$H_{0,0} := \left[\frac{1}{(\pi)^{.5}}\right]\cdot\Gamma\left[\frac{N+1}{2}\right] \qquad H_{0,1} := \left[\frac{1}{2\cdot(6)^{.5}}\right]\cdot(N\cdot(N-2))\cdot H_{0,0}$$

$$H_{1,0} := H_{0,1} \qquad H_{1,1} := \left[\frac{1}{24}\right]\cdot\left[N^4 + 4\cdot N^3 + 20\cdot N^2 + 32\cdot N + 24\right]\cdot H_{0,0}$$

$$H_{2,0} := \left[\frac{1}{(8!)^{.5}}\right]\cdot N\cdot\left[N^3 - 12\cdot N^2 + 44\cdot N - 48\right]\cdot H_{0,0} \qquad H_{0,2} := H_{2,0}$$

$$H_{1,2} := \left[\frac{1}{(4!\cdot 8!)^{.5}}\right]\cdot N\cdot\left[N^5 + 2\cdot N^4 + 36\cdot N^3 - 8\cdot N^2 - 16\cdot N - 288\right]\cdot H_{0,0}$$

$$H_{2,1} := H_{1,2} \qquad H_{2,2} := \left[\frac{N^4}{8!}\right]\cdot\left[\begin{array}{l} N^4 + 8\cdot N^3 + 168\cdot N^2 + 896\cdot N \ldots \\ + 6384 + 19712\cdot N^{-1} + 52352\cdot N^{-2} \ldots \\ + 67584\cdot N^{-3} + 40320\cdot N^{-4} \end{array}\right]\cdot H_{0,0}$$

2. CALCULATE LOWEST EIGENVALUE

Guess at eigenvalue: $x := 10^{-6}$ Set solution tolerance: $TOL := 10^{-7}$

Solve for lowest eigenvalue: $\lambda := root(|H - x\cdot identity(3)|, x)$

For $N = 5$ the 3-dimensional solution is: $\lambda = 0.9754$

The ground state value is: $s := \left[\left[\pi^{-.5}\right]\right]\cdot\Gamma\left[\frac{N+1}{2}\right]$. Value of $s = 1.1284$

Figure 2. Statement of MathCAD program.

so that for instance

$$\left\langle 0\left|\frac{|x|^2}{2}\right|2\right\rangle = -\left\langle 0\left|\frac{|p|^2}{2}\right|2\right\rangle.$$

In other words

$$\langle 0|H|2\rangle = 0$$

i.e. the state $|2\rangle$ (and similarly $|6\rangle$, $|10\rangle\ldots$) does not connect with $|0\rangle$. On the other hand, for instance

$$\left\langle 0\left|\frac{|x|^2}{2}\right|0\right\rangle = \left\langle 0\left|\frac{|p|^2}{2}\right|0\right\rangle$$

i.e.

$$\langle 0|H|0\rangle = 2\left\langle 0\left|\frac{|x|^2}{2}\right|0\right\rangle$$

while

$$\left\langle 0\left|\frac{|x|^2}{2}\right|4\right\rangle = \left\langle 0\left|\frac{|p|^2}{2}\right|4\right\rangle$$

i.e.

$$\langle 0|H|4\rangle = 2\left\langle 0\left|\frac{|x|^2}{2}\right|4\right\rangle$$

etc.

It is elementary to find the matrix elements $H_{0,0}$, $H_{0,1}=H_{1,0}$, $H_{1,1}$, where 0, 1, 2, ... correspond to $|0\rangle$, $|4\rangle$, $|8\rangle$.... These are shown in figure 2. For the 2×2 space one then solves the characteristic equation, and the exact solution for λ is:

$$\lambda = \left[\frac{\Gamma(N)/2^{N-1}\Gamma(N/2)}{48}\right][f-[f^2-768[N^3+2N^2+4N+3]]^{1/2}] \qquad (7)$$

where

$$f=[N^4+4N^3+20N^2+32N+48].$$

5. Approximate solution in subspace spanned by $|0\rangle$, $|4\rangle$, $|8\rangle$

For this 3×3 space we need the additional matrix elements $H_{0,2}=H_{2,0}$, $H_{1,2}=H_{2,1}$, and $H_{2,2}$. These are also given in figure 2, where an N^4 factor is removed from the $H_{2,2}$ element to improve the convergence when using the approximation scheme.

To solve this problem using the MathCAD 'program' shown in figure 2, we first choose a particular N, then set up the corresponding 3×3 matrix and find the lowest root of the characteristic equation, which is plotted in figure 1 alongside the 1×1 solution.

The 3×3 solution does not differ very much from the 2×2 solution which lies between the two curves in this figure. For example for $N=5$ the 3×3 solution $=0.9754$, whereas the 2×2 solution gives 0.9864 and the ground state solution is 1.1284, i.e. the 2×2 approximation is ~87.4% the ground state solution while the 3×3 approximation

is ~86.4% the ground state solution for this value of N. From our previous work we have found the general result that increasing the space beyond 3×3 does not significantly alter the expectation value. Actually for $N=4, 6, 8$ in the 3×3 space we obtain 0.6984, 1.4822 and 4.2402 respectively versus the essentially exact results of [1] namely 0.6984, 1.4765, and 4.1447 for these three N's. Thus we expect the exact ground state of H for positive N to be quite close to the values we obtain for the 3×3 case.

For negative N, however, one observes a discontinuous derivative and an abrupt decrease in the value of λ as we cross the value $N=0$. This result we interpret as being due to the fact that for $N<0$ the variational technique is trying to admix higher states (which have smaller components at the origin) into the wavefunction in order to counter the singularity of the $|x|^N$, $|p|^N$ operators at the origin. Another way of viewing this result for negative N is by noting that whereas $H_{1,1}$, $H_{2,2}$ are positive for positive N, with $H_{1,1}<H_{2,2}$, both these matrix elements become negative for N less than zero with $H_{2,2}$ in fact crossing and becoming more negative than $H_{1,1}$ as N approaches -1.

6. Summary and discussion

A generalization of the ordinary mean-squared fluctuation of an operator used in ordinary quantum mechanics, utilizing the absolute value, allows one to discuss the minimum uncertainty product of two canonically conjugate operators for any value of $N>-1$. For $N>0$ the authors' approximation scheme yields values which are expected to be very close to that which would be obtained by an exact calculation.

For negative $N>-1$ the approximation scheme essentially fails and the authors offer a qualitative explanation for this development.

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